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**Frege’s Result: Frege’s Theorem and Related Matters**

**Abstract** One of the remarkable results of Frege’s Logicism is Frege’s Theorem, which holds that one can derive the main truths of Peano arithmetic from Hume’s Principle (HP) without using Frege’s Basic Law V. This result was rediscovered by the Neo-Fregeans and their allies. However, when applied in developing a more advanced theory of mathematics, their fundamental principles—the abstraction principles—incur some problems, e.g., that of inflation. This paper finds alternative paths for such inquiry in extensionalism and object theory.

**Keywords** Frege’s Theorem, Neo-Fregeans, abstraction principles, extensionalism, object theory

**1 Introduction**

The purpose of this paper is to present an elementary explanation of Frege’s remarkable result, “Frege’s Theorem,” by G. Boolos, to examine its philosophical significance and problems, and further to research on alternative studies pertaining to “Neologicism” (or the “Neo-Fregean program”).

J. Burgess presents the history of work on Frege’s Theorem (Burgess 2005, 147–49). C. Parsons found that “working from Hume’s Principle (HP) one can derive arithmetic” (qtd. in Buegess 2005, 147); on the other hand, P. Geach observed that “the Russell paradox does not arise if one drops [Basic] Law V (Axiom V) and works only from Hume’s Principle” (Geach 1975, 446–47).

In 1983 C. Wright’s book, *Frege’s Conception of Numbers as Objects*, “for the first time brought together [the] two observations” (Burgess 2005, 147). The book treated both the possibility of deriving arithmetic (Peano axioms) from Hume’s Principle (HP) and of showing the consistency of HP. He did not, however, complete the proofs, but only giving a sketch of the derivation of the
Peano axioms and offering the conjecture that the system with HP not containing Basic Law V is consistent. After that, Boolos—who introduced the terms “Frege’s Theorem,” “Hume’s Principle,” and “Frege Arithmetic (FA)” —explicitly showed not only that the Peano axioms can be derived in FA, but also that FA is interpretable in second order Peano arithmetic. Many scholars working on these matters today take Boolos’s work as their starting point. I will touch on a few of these studies in the last two sections.

2 The Heritage of Begriffsschrift

As a preparation for the discussion about deriving arithmetic from HP and second order logic, we must first look back at some definitions and concepts provided in Frege’s first book, Begriffsschrift (Conceptual Notation). Following Frege’s original method, one can carry out the derivation, using only the means which occur in his Die Grundlagen der Arithmetik (The Foundations of Arithmetic), which takes over many logical definitions and concepts from Begriffsschrift.

Begriffsschrift part III contains the following four definitions. For ease of comprehension I will use a parent-child metaphor here for explaining the $f$-sequence (f-relation, or f-procedure). If $xfy$, an object $x$ stands in $f$-relation to an (other) object $y$, and I say “$x$ is a parent of $y,” or “$y$ is a child of $x.” Further, I call “family” a chain consisting of repeated $f$-relations.

Definition 1  Inheritance of $F$:
A property $F$ is inherited in the $f$-sequence (or the $f$-relation), if and only if,

\[
\forall x\forall y((Fx \land xfy) \rightarrow Fy).
\]

(“Her” comes from “hereditary.”)

Definition 2  Proper ancestry:
An object $x$ is an ancestor of an object $y$ (an object $x$ precedes an object $y$ in the $f$-sequence; $y$ follows $x$ in the $f$-relation), if and only if,

\[
xf^*y := \forall F[(Her(F) \land \forall z(xfz \rightarrow Fz)) \rightarrow Fy].
\]

Definition 3  Belonging to a family:
An object $y$ belongs to the $f$-family ($f$-sequence) beginning from $x$,
if and only if,

an object \( x \) is an ancestor of an object \( y \) or \( y \) is the same as \( x \).

\[
x f^\ast y := x f^\ast y \lor y = x.
\]

**Definition 4** Uniqueness:

A relation \( f \) is many-one (or a function, a result of applying the procedure \( f \) to some object is uniquely given), if and only if,

for any object \( x, y, z \), if \( y \) is a result of applying the procedure \( f \) to \( x \) and \( z \) is also a result of applying the procedure \( f \) to \( x \), then \( y \) is the same as \( z \).

\[
FN(f) := \forall x \forall y \forall z ((x f y \land x f z) \rightarrow y = z).
\]

(“FN” comes from “function.”)

Definition 1 expects \( F \) to be a property of natural numbers, for example, \( 0+1+2+\cdots+n = 1/2 \cdot n(n+1) \) is inherited in the sequence of natural numbers. I show the meaning can be shown by drawing a picture. If we have a \( f \)-sequence (or a chain of \( f \)-relations) like this:

\[
\ldots \circ \circ \circ \circ \circ \circ \ldots
\]

\[
\alpha \beta \gamma \delta
\]

and if a member \( \alpha \) has \( F \): ●, which is inherited in this sequence:

\[
\ldots \circ \circ \circ \circ \circ \circ \circ \circ \ldots
\]

\[
\alpha
\]

then \( F \) is inherited from \( \alpha \) to \( \beta, \gamma, \delta \) etc. like this:

\[
\ldots \circ \circ \circ \circ \circ \circ \circ \circ \ldots
\]

\[
\alpha \beta
\]

\[
\ldots \circ \circ \circ \circ \circ \circ \circ \circ \ldots
\]

\[
\alpha \beta \gamma
\]

\[
\ldots \circ \circ \circ \circ \circ \circ \circ \circ \ldots
\]

\[
\alpha \beta \gamma \delta
\]

In this stage the \( f \)-sequence can contain the branching or the joining, though the sequence of natural numbers has a linear order and thus does not contain those structures.

Definition 2 defines the relation of an ancestor to his descendant (in an \( f \)-family). If someone has all the hereditary properties which were shared by all
the children of Genghis Khan, then he is one of Genghis Khan’s descendants, since among the properties inherited from his ancestors there is certainly contained the special property which is inherited from Genghis Khan by all and only the descendants of Genghis Khan. This definition foresees the relation between the first natural number zero, the ancestor, and other natural numbers, the descendants of zero.

Definition 3 foresees the property of one number’s belonging to the natural numbers family by stipulating that it is a descendant of zero as the ancestor, or is itself zero, the ancestor itself. Definition 4, by giving the uniqueness of some general procedure’s results, foresees the unique successor of a natural number, that is, the many-one relation of the successor in the natural numbers.

Using these definitions, Frege derived many formulas in *Begriffsschrift*. For example, he derived the formula numbered 81 in *Begriffsschrift* (III):

\[(81) \quad Fx \rightarrow [Her(F) \rightarrow (xf^+y \rightarrow Fy)].\]

From this we can derive the next formula 81*:

\[(81)^* \quad Fx \rightarrow [Her(F) \rightarrow ((xf^+y \vee y = x) \rightarrow Fy)].\]

If we interpret “x” as “0,” “xfy” as “y is the successor of x,” and “0f*y” as “y is arrived from 0 through the successor relation,” then we can read “Her(F)” as “F is inherited in the successor relation.” Therefore, by defining “y is a natural number” as “0f*y ∨ y = 0,” we get the principle of Mathematical Induction (MI):

\[(MI) \quad [F0 \land \forall n(Fn \rightarrow Fn + 1)] \rightarrow \forall nFn.\]

3 The Definition of Number and Russell’s Paradox

Five years after the publication of *Begriffsschrift*, Frege wrote *Die Grundlagen der Arithmetik* (GLA), and there he showed the program of “logicism,” i.e., of deriving arithmetic from logic. One of his most important problems in carrying out the logicist program was giving the definition of number. Frege noticed that a sentence containing expressions of numbers tells about (first order) concepts. Surely numbers are, in a certain way, connected to concepts. However, numbers cannot be higher order concepts, because one has to regard numbers as objects when he does arithmetic.

But what is the criterion of identity for numbers? This is the first problem to solve, since one can proceed to inquire into the essence of numbers only if he holds the criterion of identity for numbers and can distinguish individual numbers. His critical notion for obtaining the criterion was that of equinumerosity (*gleichzahligkeit*) between concepts connecting to numbers.
Frege’s thinking is as follows. As straight lines $x$ and $y$ have the same direction when they are parallel to each other, so too the number of $F$s is equal to the number of $G$s when the concept $F$ is equinumerous with the concept $G$. For example, the concept “prime number less than 7” is equinumerous with the concept “planet closer to the sun than Mars,” so that the number of prime numbers less than 7 is the same as the number of planets closer to the sun than Mars, that is, 3.

Thus, we get the following principle:

**Hume’s Principle (HP):**

The number of $F$s is the same as the number of $G$s if and only if the concept $F$ is equinumerous with the concept $G$
or in symbols

$$(HP) \quad \#F = \#G \leftrightarrow F \approx G.$$  

(We occasionally omit universal quantifiers that would be located at the head of sentences.)

This principle is now called “Hume’s Principle (HP).” What is equinumerosity? A concept $F$ is equinumerous with a concept $G$ if and only if there is a relation that corresponds one to one between $F$s and $G$s, between objects belonging to each concept. In symbols,

$$(\approx) \quad F \approx G \iff \exists \phi [\forall x \{Fx \to \exists y(Gy \land \forall w(x \phi w \equiv w = y)\}] \land 
\forall y \{Gy \to \exists x(Fx \land \forall u(u \phi y \equiv u = x)\}).$$

We can thus secure the identity condition of numbers. However, what is the number 3? What is number, what is the essence, if there is any such thing, of numbers? Frege’s answer is as follows.

**Definition of number:**

The number of $F$s is the extension of the (second order) concept “equinumerous with $F$” (Frege 1884, § 68). In symbols,

$$(\text{Number}) \quad \#F := \{X(F \approx X)\}.$$  

By using the extension of concepts, Frege gives the definitions of individual numbers:
the number 0 := the extension of the concept “not identical with itself
\((x\neq x)\),"
the number 1 := the extension of the concept “the same as 0 \((x=0)\),”
the number 2 := the extension of the concept “the same as 0 or 1 \((x=0 \text{ or } 1)\),”
etc.

The notion one needs for producing numbers larger than zero is that of
“successor.” It is defined as follows:

A number \(n\) is the successor of a number \(m\),
if and only if
there is a concept \(F\) and an object \(y\) such that the number of \(F\) is \(n\) and \(y\) is
\(F\) and the number of the concept “\(F\) but not the same as \(y\)” is \(m\).

In symbols:

\[ (\text{Successor}) \quad mPn \quad F \quad y \quad F \quad n \quad Fy \quad x \quad Fx \quad x \quad y \quad m \]

\[ (\text{where the concept “} F \text{ but not the same as } y \text{” is expressed as} \quad [x \quad := \quad Fx \quad \land \quad x \quad \neq \quad y]). \]

As Frege foresees, one can show the infinity of natural numbers 0, 1, 2, 3, … by
proving that any natural number has its successor (different from itself) (Frege
1884, §§ 82–83).

However the principle regarding the extensions of concepts is Basic Law V:

**Basic Law V:**

The extension of a concept \(F\) is the same as the extension of a concept \(G\)
if and only if
all \(Fs\) are \(Gs\) and vice versa (\(F\) is coextensive with \(G\)).

In symbols:

\[ \forall x (Fx \equiv Gx). \]

Frege explicitly formulated Basic Law V for the first time in his *Grundgesetze
der Arithmetik*. Of course he did not know that a contradiction (i.e., Russell’s
paradox) could be derived from Basic Law V in his system when he wrote *GLA*.

The definition of number, (second order) Basic Law V and HP are related to one
other as follows:
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Basic Law V

\[ X(F \approx X) = X(G \approx X) \iff \forall X(F \approx X \equiv G \approx X) \]

the definition of number \( \Rightarrow \)

\[ \#F = \#G \iff F \approx G \]

HP (Hume’s Principle)

Frege knew that HP was derived from Basic Law V when he wrote GLA. Moreover, he does not use Basic Law V for deriving arithmetic both in GLA and in Grundgesetze. It was this fact that the Neo-Fregeans and their allies, Parsons, Wright, and Boolos, among others, discovered.

4 A Derivation of Main Arithmetical Theorems

In this section, I define Frege Arithmetic (FA), which was introduced by Boolos (1987, 1998), and I show the consistency of HP relative to second order arithmetic and I also present the main arithmetical theorems in FA.

4.1 A Definition of FA

FA is based on binary second order logic with HP as the sole non-logical axiom, and has about the same deductive power as the relevant part of GLA, enough to derive main arithmetical theorems. FA has three kinds of variables:

1. Object variables: \( a, b, c, d, m, n, x, y, z, \ldots \)
2. One-place predicate variables: \( F, G, H, \ldots \)
3. Two-place predicate variables: \( \phi, \psi, \chi, \ldots \)

Each variable ranges over, respectively, objects, first level concepts, and first level binary relations. As the sole non-logical symbol of FA, we introduce the two-place predicate symbol “\( \eta \).” I write:

\[ Fx, \]

read “a concept \( F \) belongs to the extension \( x \).” The atomic formulas of FA are \( Fx, x\eta y, \) and \( Fx. \) Following Leibniz and Frege, the identity of objects, \( x = y, \) is defined as \( \forall x(Fx \equiv Gx) \). We adopt the usual axioms of second order logic, e.g., the comprehension axioms.

As the sole non-logical axiom of FA, we introduce the formula called “Numbers” (Boolos 1998, 186):

\[ (\text{Numbers}) \quad \forall F \exists ! x \forall G(G \eta x \equiv F \equiv G). \]

The axiom “Numbers” claims that for any first level concept \( F \), there is the
unique object $x$ which corresponds to the extension of $F$ such that any first level concept $G$ belongs to $x$ if and only if $G$ is equinumerous with $F$. In FA, the axiom of Numbers is provably equivalent to HP:

$$\text{(HP)} \quad \#F = \#G \leftrightarrow F \approx G.$$ 

That is, one can derive HP from the axiom Numbers in FA (Tabata 2000),

$$\text{Numbers} \vdash_{FA} \text{HP}.$$ 

And further, the axiom Numbers is derived from HP in FA (Tabata 2000),

$$\text{HP} \vdash_{FA} \text{Numbers}.$$ 

### 4.2 The Consistency of HP

I can now show the consistency of HP. I give the following model $\mathcal{M} = \langle U, \sigma \rangle$. The domain of $\mathcal{M}$ is $U$. $U$ consists of all natural numbers including 0 and $\aleph_0$ (aleph zero):

$$U = \{0, 1, 2, 3, \ldots, \aleph_0\}.$$ 

The valuation function $\sigma$ gives an object variable an object in $U$, a one-place predicate variable a subset of $U$, and a two-place predicate variable a subset of $U^2$. To sum up:

$$\sigma(a) \in U$$

$$\sigma(F) \in \wp(U) = \{V : V \subseteq U\}$$

$$\sigma(\phi) \in \wp(U^2) = \{V : V \subseteq U^2\}.$$ 

In this model $\mathcal{M} = \langle U, \sigma \rangle$, we interpret “#” as a function: $\wp(U) \to \{ |V| : V \subseteq \wp(U) \}$, that is: the function which produces the cardinal number $|V|$ of $V$ as an output, if given a subset $V$ of $U$ as an input. Then,

$$\sigma(#F = #G) = T (true) \iff \sigma(#(F)) = \sigma(#(G))$$

$$\iff \#(\sigma(F)) = \#(\sigma(G))$$

$$\iff \exists f (\sigma(F) \text{ corresponds one-to-one to } \sigma(G) \text{ by } f)$$

$$\iff \sigma(F \approx G) = T ,$$

thus, $\sigma(#F = #G) = T \iff \sigma(F \approx G) = T$,

therefore, $\sigma(#F = #G \leftrightarrow F \approx G) = T$.

Thus HP is satisfied in this model and it is consistent.

### 4.3 Arithmetical Theorems

Now I present the main arithmetical theorems which are derived in FA, the
system of second order logic with HP as an axiom. Among the theorems derived are contained the five axioms of second order Peano arithmetic. In the following, “Num” means “natural number” and “xPy” means “x precedes y” or “y is the successor of x.”

**Axiom 1**  Zero is a natural number: Num 0.

**Axiom 2**  Every natural number has a unique successor, which is also a natural number:

\[ \forall x \ (\text{Num } x \rightarrow \exists y \ (\text{Num } y \land xPy \land \forall z (xPz \rightarrow z = y))) \].

**Axiom 3**  Zero is not the successor of any natural number:

\[ \forall x \ (\text{Num } x \rightarrow \neg xP0) \].

**Axiom 4**  For any natural number x, y, if the successor of x is identified with the successor of y, then x is the same as y:

\[ \forall x \forall y \forall z \ [(\text{Num } x \land \text{Num } y \land \text{Num } z \land xPz \land yPz) \rightarrow x = y] \].

**Axiom 5**  For any property F, if zero has F, and every successor of a natural number which has F also has F, then every natural number has F:

\[ \forall F [\{ F0 \land \forall x \forall y (Fx \land xPy \rightarrow Fy) \} \rightarrow \forall x (\text{Num } x \rightarrow Fx)] \].

As I have already shown the details of the derivation (Tabata 2000, 2002), here I do not give complete proofs but rather present the main theorems. The above five axioms are contained in the following theorems.

**Theorem 1**  \( \forall F (\# F = 0 \leftrightarrow \forall x \neg Fx) \).

(The number belonging to a concept F is zero iff no object has F.)

**Theorem 2**  \( \forall m \forall n [mPn \land m' Pn' \rightarrow (m = m' \leftrightarrow n = n')] \).

(For any object m, n, if n is the successor of m and n’ is the successor of m’, m is equal to m’ iff n is equal to n’.)

**Corollary 1**  (Peano’s fourth axiom)  \( \forall x \forall y \forall z [(\text{Num } x \land \text{Num } y \land \text{Num } z \land xPz \land yPz) \rightarrow x = y] \).

**Theorem 3**  \( \forall x \neg xP0 \).

(There is no predecessor of zero.)

**Corollary 2**  (Peano’s third axiom)  \( \forall x (\text{Num } x \rightarrow \neg xP0) \).

Here, I use the definition 2 of “proper ancestry” in §2 with “R” for “f.”

**Definition 2**  \( xR^* y := \forall F [(\text{Her}(F) \land \forall z (xRz \rightarrow Fz)) \rightarrow Fy] \).

**Theorem 4**  \( \forall x \forall y (xRy \rightarrow xR^* y) \).

(For any two objects, if they stand in the “parent-to-child” relation, they also stand in the “ancestor-to-descendant” relation.)

**Theorem 5**  (Transitivity of \( R^* \))  \( \forall x \forall y \forall z (xR^* y \land yR^* z \rightarrow xR^* z) \).
Now we apply $R^*$, which is the relation of “ancestor-to-descendant,” to $P^*$, which is the relation of “following after” in the natural number sequence.

Theorem 6 \(\forall x \forall n [x P^* n \rightarrow \exists m P n \land \exists m P n \land (x P^* m \lor x = m)]\).

(For any object $x$ and $n$, if $n$ follows $x$ (in the natural number sequence), then there exists some predecessor of $n$, and every predecessor $m$ of $n$ follows $x$ or is equal to $x$.)

Theorem 7 \(\forall n (0 P^* n \rightarrow \neg n P^* n)\).

(No number which follows after $0$ follows after itself.)

Definition 5 (Definition of the relation of smaller-to-greater)

\[ m \leq n := m P^* n \lor m = n. \]

Definition 6 (Definition of natural (=finite cardinal) number)

\[ \text{Num } n := 0 \leq n. \]

Theorem 8 (Peano’s first axiom) Num 0.

Theorem 9 (Mathematical Induction: Peano’s fifth axiom)

\[ \forall F[\{F 0 \land \forall x \forall y ((F x \land x P y) \rightarrow F y) \rightarrow \forall x (\text{Num } x \rightarrow F x)]]. \]

Theorem 10 \(\forall m \forall n \forall \eta ([m P n \land 0 P^* n] \rightarrow \forall x (x \equiv m \leftrightarrow (x \equiv n \land x \neq n))]\).

(For any $m$, $n$, if $m$ is the predecessor of $n$ and $n$ follows after 0 in the natural number sequence, then every natural number is smaller than $m$ or equal to $m$ iff it is smaller than $n$.)

Theorem 11 \(\forall m \forall n ([m P n \land 0 P^* n] \rightarrow \# [x : x \leq m] P \# [x : x \equiv n])\).

(For any $m$, $n$, if $m$ is the predecessor of $n$ which follows after 0 in the sequence of natural numbers, then the number which belongs to the concept “smaller than $m$ or identical with $m$” is the predecessor of the number which belongs to the concept “smaller than $n$ or identical with $n.”)

Theorem 12 \(\forall m \forall n [m P n \rightarrow \{(0 \leq m \land m P \# [x : x \leq m]) \rightarrow 0 \equiv n \land n P \# [x : x \equiv n]\}]\).

(For any $m$, $n$, if $m$ is the predecessor of $n$, then, if 0 is smaller or identical with $m$ and $m$ is the predecessor of the number which belongs to the concept “smaller than or identical with $m$,” then 0 is smaller than or identical with $n$ and $n$ is the predecessor of the number which belongs to the concept “smaller than or identical with $n.”)

Theorem 13 \(0 P \# [x : x \equiv 0]\).

(0 is the predecessor of the number which belongs to the concept “smaller than or identical with 0.”)

Theorem 14 \(\forall n (0 \equiv n \leftrightarrow 0 \equiv n \land n P^* \# [x : x \equiv 0])\).

Theorem 15 \(\forall n (\text{Num } n \rightarrow n P^* \# [x : x \equiv n])\).
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Corollary 3 (Peano’s second axiom).
\[ \forall m (\text{Num } m \rightarrow \exists! n (\text{Num } n \land mPn)) . \]
(Any natural number has the unique successor of itself which is also a natural number.)

5 Examinations

Now I will briefly examine the philosophical significance of Frege’s Theorem and related matters.

5.1 Frege’s Theorem as a Starting Point

First, Frege’s Theorem shows that some subsystem of Frege’s logical system, i.e., second order logic + HP without Basic Law V (=FA), is enough to derive the arithmetic of natural numbers (i.e., finite cardinals). The only non-logical axiom, HP, keeps relative consistency. This is thus a modest success of Frege’s logicist program.

However, one will immediately ask about mathematics beyond the arithmetic of natural numbers. How can one derive the arithmetic of real numbers, or real analysis? The Neo-Fregeans and their allies, for example, Wright, Hale, and Shapiro, show how to develop higher mathematics using various abstraction principles such as HP (Shapiro 2000; Wright 2000). For example, one can use the Pairs abstraction principle:

(Pairs) \[ \forall x \forall y \forall z \forall w (< x, y > = < z, w > \leftrightarrow (x = z \land y = w)) \]
to arrive at ordered pairs of natural numbers. Then we get integers by regarding a pair \( \text{Int}(a, b) \) as an integer, using the Difference abstraction (provided that we have additions and multiplications in natural numbers):

(Difference) \[ \forall a \forall b \forall c \forall d (\text{Int}(a, b) = \text{Int}(c, d) \leftrightarrow (a + d = c + d)) . \]

We proceed to rational numbers by regarding a pair \( \text{Q}(m, n) \) of natural numbers \( m, n \) as a rational number, using the Quotient abstraction:

(Quotient) \[ \text{Q}(m, n) = \text{Q}(p, q) \leftrightarrow [n = 0 \land q = 0) \lor (n \neq 0 \land q \neq 0 \land mq = np)] . \]

Thus we arrive at real numbers, if \( P \) is bounded and not empty, by regarding a Cut\((P)\) as a real number, using the Cut abstraction from Dedekind’s idea:
(Cut) \( \forall P \forall Q (\text{Cut}(P) \leftrightarrow \forall r (P \leq r \equiv Q \leq r)) \),

where \( P, Q \) are properties (not sets) of rational numbers, and “\( P \leq r \)” means that a rational number \( r \) is an upper boundary of \( P \), that is, \( r \) is greater than or equal to any rational number \( s \) to which \( P \) applies. FA (i.e., second order logic +HP) plays a crucial role as the basis of this development. Thus, here too Frege’s Theorem has become a starting point of developing other systems than those of natural numbers.

5.2 Abstract Principles

Second, we think of abstraction principles. Hume’s Principle

\[(\text{HP}) \quad \# F = \# G \leftrightarrow F \equiv G\]

gives a contextual definition of (cardinal) numbers. Of course, in the full system of Grundgesetze, Frege gave the definition of numbers using the extensions of higher order concepts:

\[(\text{Number}) \quad \# F := \#X(F \approx X).\]

(The number of \( F \)'s is the extension of the concept “equinumerous with \( F \).”)

This definition works with the second order version \( \forall x (F \approx X) = \forall x (G \approx X) \leftrightarrow \forall x (F \equiv G \equiv X) \) of (original) Basic Law V:

\[(\text{Basic Law V}) \quad \forall x (F \equiv G \equiv X) = \forall x (F \equiv G \equiv X).\]

The common feature between HP and Basic Law V is that they are both abstraction principles. In general, we can understand the abstraction principle “Abstraction” as follows:

\[(\text{Abstraction}) \quad \forall a \forall b (\Sigma(a) = \Sigma(b) \leftrightarrow E(a,b)),\]

where \( a, b \) are variables of given-type items (i.e., objects or properties), and \( \Sigma \) means a higher order function from given-type items to the range of items of first order variables, and \( E \) is an equivalence relation over items.

There are some problems with abstraction principles. One of them is that of inflation. If the intended interpretation of the base theory is finite, of size \( n \), then HP produces the existence of \( n+1 \) cardinal numbers. Suppose the original domain of objects is \( \{a_1, a_2, a_3\} \), so \( n = 3 \). Then the cardinal numbers produced by HP are

\[
\begin{align*}
0 &= \# \emptyset \\
1 &= \# \{a_1\} = \# \{a_2\} = \# \{a_3\}, \\
2 &= \# \{a_1, a_2\} = \# \{a_1, a_3\} = \# \{a_2, a_3\}, \\
3 &= \# \{a_1, a_2, a_3\},
\end{align*}
\]

so \( n+1 = 4 \). This is a mild inflation. However, as the quantifiers in HP are not restricted, the principle entails the existence of numbers, i.e., \( n+2 \), of properties of those cardinal numbers. But this inflation ends, since the result of adding natural numbers and \( \aleph_0 \) to the domain of the original model makes the structure which satisfies HP. But Basic Law V, beginning from \( n \) items in the domain, produces \( 2^n \) extensions. And further, it produces extensions of properties of extensions of original objects in the domain. Thus this inflation does not end. Scholars are currently engaged in extracting the characteristics from inflations, classifying them and presenting criteria for accepting them (see Cook 2002; Shapiro 2000).

5.3 Second Order Logic and Logicism

Here I consider second order logic and logicism. Quine, an influential leader who once brought prosperity to American analytic philosophy, interfered with the development of at least two parts of logic, modal logic and second order (higher order) logic. He doubted the possibility of understanding modal logic, but modal logic has now formed a major field of logical research. For example, provability logic, which is a version of modal logic and is concerned with Gödel’s second incompleteness theorem, has presented a meta-logical analysis of provability predicates in Peano arithmetic (see Boolos 1993). As for second order logic, Quine called it “set theory in sheep’s clothing” (Quine 1986, ch.5; Shapiro 1991, 194). He played favorites with first order logic, since he regarded the quantification of predicates as that of attributes and relations and hated entities such he recognised as intentional. Quine criticized Carnap, but himself retains still bad a nominalistic tendency of positivism.

Frege made use of second order logic with extensions of concepts (value-ranges of functions), and expected logic or logical concepts to be an epistemological base. Frege held that the validity of arithmetical proofs originates from that of the principles of logic. He regarded proofs as a process of following the line of validity toward the laws of logic as ultimate grounds. But what is logic? We cannot a priori characterize logic by means of any property, e.g., certainty, analyticity, topic-neutrality, or ontological transparency. The significance of Frege’s Logicism which we should adopt is not that of giving logical systems the holy title “logic” based on some characterization, but rather that of investigating the systems’ range of deductive power as far as possible. (“Reverse Mathematics”—the study concerning which axioms each mathematical theorem needs to be proved—which uses second order arithmetic, is a promising challenge to the spirit of Frege’s Logicism.) The Neo-Fregeans’ Neologicism is one version of an experiment which inherits Frege’s ideas.

In any event, Frege’s Theorem gives us a chance to think about various logical
themes, so in that sense, it is a starting point for logical research. However, some authors present alternatives to the Neo-Fregean’s ways of reviving Frege. In the last section I touch on two of those lines of inquiry. One is “extensionalism” and the other is “object theory.”

6 Alternatives

In this last section I glance at alternatives to Neo-Fregeans’ Logicism.

6.1 Extensionalism

In Antonelli and May (2005) we can find an example of extensionalism as an alternative program to that of the Neo-Fregeans. They claim that Frege’s program consists of two parts, “logicism,” in Grundlagen, and “extensionalism,” in Grundgesetze. The first part, as they see it, is the idea that arithmetic should be reducible to logic, and the second is the idea that arithmetic should be recovered from a theory of extensions. According to Antonelli and May, “Frege’s work lies at the intersection of these two programs, and it amounts to the idea that arithmetic should be reconstructed in terms of a logical theory of extensions” (Antonelli and May 2005, 1). They criticize the strategy of the Neo-Fregeans (Wright and Hale) which rejects Frege’s Basic Law V, contending that if we want to get such a strong system as the Peano axioms enough to develop arithmetic rather than such a weak system as Robinson’s Q, we must hold the extensions of concepts (Frege’s value ranges). They present a formal system $ℱ$ and its interpretation. $ℱ$ is a second order system with a second order comprehension axiom and a version of Frege’s Basic Law V (which has a partial mapping in order to avoid the contradiction with Cantor’s theorem), as well as an extra-logical axiom schema ensuring the existence of certain value ranges (Antonelli and May 2005, § 3). They construct numbers not as objects but as concepts. This is the distinguishing feature of their “extensionalism,” in contrast to the way of the Neo-Fregeans. They develop, within their system, fundamental theorems of arithmetic including mathematical induction and other Peano’s axioms (see Antonelli and May 2005, 6–11). The consistency of $ℱ$ is also secured (Antonelli and May 2005, § 4). Further, they show a counterexample of Hume’s Principle from the standpoint of their own system. But it is not easy to get a perspective of developing arithmetic in their ways because of their treating numbers as concepts not as objects. Although I cannot here go into the details of their argument, I think their project forms a strong alternative to that of the Neo-Fregeans.
6.2 Object Theory

Another alternative to the Neo-Fregeian’s program I consider is that of object theory. We can find an example of it in Zalta (1999) and in Linsky and Zalta (2006). In the former study, Zalta (1999) presents a general theory of abstract objects formulated in second order $S5$ modal predicate calculus, modified so as to include $xF$ (“$x$ encoded $F$”) as an atomic formula along with $F^n x_1 \ldots x_n$. He gives some modal formulas as proper axioms which, he holds, contain only logical notions (rather than mathematical ones such as “number”), and he shows that Peano axioms and the main theorems of arithmetic are derived from that system. One of the features of this system is that it distinguishes ordinary (concrete) objects from abstract objects, with the result that we can solve the “Caesar problem”; and another feature of it is, as Zalta emphasizes, the naturalness (logicalness) of the way he constructs his system.

In the second study, Linsky and Zalta (2006) present a third order non-modal object theory as the best way to reconstruct logicism. In their paper, they reconsider logicism and “neologicism,” trying to chart the terrain of “a variety of positions that might properly be called ‘neologicism’” (Linsky and Zalta 2006, 60) and claiming that “[a] view which might be called ‘neologicism’ can be produced by (any combination of) the following three strategies:

1. Expand the conception of what counts as ‘logic’
2. Allow more resources than ‘logic alone’
3. Reconceive the notion of ‘reducible’” (Linsky and Zalta 2006, 64).

After having investigated these strategies and related approaches (Fine’s abstraction theory and Boolos’s New V), they conclude that “Third-order object theory is a neologicism because it reduces […] all of mathematics to third-order logic and some analytic truths” (Linsky and Zalta 2006, 91).

Thus, their study presents a reconsideration of and a broad range of perspectives on “neologicism,” and at the same time gives a program alternative to that of the Neo-Fregeans (Wright and Hale 2001).

To repeat, Frege’s Theory, which has its origin in Frege’s own work and has been spotlighted anew by the Neo-Fregeans and their allies, leads us to stimulating research in logic and mathematics.

References


